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PROJECTIVE INJECTIVE AND FLAT MODULES OVER UPPER TRIANGULAR MATRIX ARTIAN ALGEBRAS

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ABSTRACT:

In this article we determine all the n- Strongly Complete Projective Injective and Flat resolutions and all the n-Strongly Gorenstein Projective, Injective and Flat Modules over upper Triangular Matrix artin algebras.

KEYWORDS: *Gorenstein Projective, Injective and Flat Modules, Strongly Gorenstein Projective, Injective and Flat Modules, n- Strongly Gorenstein Projective, Injective and Flat Modules, Upper Triangular Matrix artin algebras.*

1. INTRODUCTION

Since Eilenberg and Moore [1] explained the relative homological algebra, especially the Gorenstein homological algebra, has been developed to an advances level: the analogues for projective, injective and flat modules are respectively the Gorenstein projective, injective and flat modules introduced by Enochs and Jenda [2] .

Throughout this article R is a commutative ring with unit element, and all R modules are unital. If M is any R-Module, we use $pd_R(M)$, $id_R(M)$ and $fd_R(M)$ to denote the usual projective, injective and flat dimensions of M, resp.

Auslander and Bridger [3] introduced the G dimension for finitely generated modules over Noetherian rings in 1967-69 denoted by G-dim(M) where G-dim(M)$\leq$pd(M), G-dim(M)$\leq$ id(M) and G-dim(M)$\leq$fd(M). If G-dim(M)=pd(M)=id(M)=fd(M) then it is finite.

The Gorenstein projective, injective and flat dimension of a module [4] is defined in terms of resolutions by Gorenstein projective, injective and flat modules respectively.

Definition:

1. An R-mod M is said to b G-projective (Short of Gorenstein projective) if there exists an exact sequence of projective modules
   P = ...$\rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$ such that M $\cong Im(P_0 \rightarrow P^0)$ and such that Hom_R(- ,Q) leaes the sequence P exact whenever Q is a projective module. The exact sequence P is called a complete projective resolution.

2. An R-mod M is said to b G-injective (Short of Gorenstein injective) if there exists an exact sequence of projective modules
   ......$\rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots = I$such that M $\cong Im(I_0 \rightarrow I^0)$ and such that Hom_R(Q,-) leaves the sequence I exact whenever Q is a injective module. The exact sequence I is called a complete projective resolution.
3. An R-mod M is said to b G-flat (Short of Gorenstein flat) if there exists an exact sequence of projective modules

F= ......$\rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$ such that M $\cong Im(F_0 \rightarrow F^0)$ and such that - $\otimes I$ leaves the sequence F exact whenever I is a injective module. The exact sequence F is called a complete Flat resolution.

2. STRONGLY GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT MODULES:

In this section we introduce and study the strongly Gorenstein projective injective and flat modules which are defined as follows [5]:

Definition: A complete projective resolution of the form

P = ...$\xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \dots$.. is called strongly complete projective resolution and denoted by (P, f).

An R-mod M is called strongly Gorenstein projective if M$\cong Kerf$ for some strongly complete projective resolution (P, f).

A complete injective resolution of the form

........$\xrightarrow{f} I \xrightarrow{f} I \xrightarrow{f} I \xrightarrow{f} \dots\dots = I$is called strongly complete injective resolution and denoted by (I, f).

An R-mod M is called strongly Gorenstein injective if M$\cong Kerf$ for some strongly complete injective resolution (I, f).

A complete flat resolution of the form

$F = \cdots.\xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \dots$.. Is called strongly complete flat resolution and denoted by (F, f).

An R-mod M is called strongly Gorenstein injective if M$\cong Kerf$ for some strongly complete flat resolution (F, f).

2.1 N- STRONGLY GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT MODULES:

Let n be a positive integer. A module M ∈ R-mod is called n-strongly Gorenstein projective if there exist an exact sequence

0$\rightarrow M \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \dots\dots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0$ in Mod R with P_i projective for 0$\leq i \leq n-1$such that Hom_R(-,P) leaves the sequence exact whenever P ∈ Mod R is projective.

Let n be a positive integer. A module M ∈ R-mod is called n-strongly Gorenstein injective if there exist an exact sequence

0$\rightarrow M \xrightarrow{f_0} I_0 \xrightarrow{f_1} \dots\dots \xrightarrow{f_{n-1}} I_{n-1} \xrightarrow{f_n} M \rightarrow 0$ in Mod R with I_i injective for 0$\leq i \leq n-1$such that Hom_R(I,-) leaves the sequence exact whenever I ∈ Mod R is injective.

Let n be a positive integer. A module M ∈ R-mod is called n-strongly Gorenstein flat if there exist an exact sequence

$0 \to M \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \ldots\ldots \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$ in Mod R with $F_i$flat for $0 \leq i \leq n-1$such that $Hom_R($ - $\otimes F)$ leaves the sequence exact whenever $P \in$ Mod R is flat.

On the basis of above following facts holds:

1. A module is Gorenstein projective (resp. Injective) if and only if it is a direct summand of a n-strongly Gorenstein projective (resp. Injective) module.
2. For finite finitistic projective dimension every n-strongly Gorenstein projective module is n-strongly Gorenstein flat module.

Proposition 1: Every projective (resp. Injective) module is n- strongly Gorenstein projective (resp. Injective)

Proof: Since every projective module is strongly Gorenstein projective then it is n-strongly Gorenstein projective (resp. Injective).

$$0 \to M \xrightarrow{f_n} P_{n-1} \oplus P_{n-1} \xrightarrow{f_{n-1}} \ldots\ldots \xrightarrow{f_1} P_0 \oplus P_0 \xrightarrow{f_0} M \to 0$$

Where M$\cong Kerf$

Consider a projective module Q applying the functor Hom R(-, Q) to the above module M for P we get the following commutative diagram:

$$\ldots\ldots\ldots\to Hom(M \oplus M, Q) \xrightarrow{Hom_R(f,Q)} Hom(M \oplus M, Q) \to\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\to Hom(M, Q) \oplus HOM(M, Q) \to Hom(M, Q) \oplus Hom(M, Q) \to\ldots\ldots.$$

The n-strongly Gorenstein projective (resp. Injective) modules are not necessarily projective (resp. Injective).

Theorem 1: A module is Gorenstein projective (resp. Injective) if and only if it is a direct summand of a n-strongly Gorenstein projective (resp. Injective) module.

Proof: Let M be a Gorenstein projective. Then there exist a complete projective resolution

$$0 \to M \xrightarrow{f_n^P} P_{n-1} \xrightarrow{f_{n-1}^P} \ldots\ldots \xrightarrow{f_1^P} P_0 \xrightarrow{f_0^P{}_0} M \to 0$$

Such that M$\cong Kerf_0^P$

Consider the exact sequence

$$0 \to \oplus M \xrightarrow{f_n^P} \oplus P_{n-1} \xrightarrow{f_{n-1}^P} \ldots\ldots\ldots\ldots \oplus P_0 \xrightarrow{f_0^P} M \to 0$$

Since Ker($\oplus f_i$)$\cong$ $\oplus$Kerf_i ,M is a direct summand of Ker($\oplus f_i$)

Moreover Hom $(\oplus_{i \in I} P_i , M) \cong \prod_{i \in I}(\oplus P_i , M)$

Which is an exact sequence for any projective (resp. Injective) module M. Thus M is a n-Strongly Gorenstein projective Module over direct summand.

2.2 N- STRONGLY PROJECTIVE, INJECTIVE AND FLAT MODULE OVER UPPER TRIANGsULAR MATRIX:

In this section determine the strongly complete projective (resp. Injective) resolutions and hence all the n-Strongly projective modules over an upper triangular matrix $\tau = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be an artin algebra of matrix.

Let X : = $\begin{pmatrix} P\oplus(M\otimes_B Q \\ Q \end{pmatrix}$, f: =$\begin{pmatrix} \alpha & 0 \\ \beta & id_M\otimes g \\ & g \end{pmatrix}$: $X\to X$ with P a projective A- module and Q a projective B- module.

Lemma: If M is an A B bimodule such that ${}_AM$ and M_B are projective modules and Hom_A(M,A) is a projective B-module or injective A-module then X is n-SG-projective (resp. injective) left B-module, then $M\otimes_B X$ is a n-SG projective A-module.

Proof: Since X is n-SG projective left module there is a complete B-projective resolution

0→ $M \overset{f_n}{\to} P_{n-1} \overset{f_{n-1}}{\longrightarrow} \dots . . \overset{f_1}{\to} P_0 \overset{f_0}{\to} M \to 0$ such that M≅ $Kerf_0$ since M_B is a projective module.

0→ $M\otimes_B P_{n-1} \xrightarrow{id_M\otimes f_{n-1}} \dots\dots\dots M\otimes_B P_0 \xrightarrow{id_M\otimes f_0} M\otimes_B P_1 \to 0$

Is exact, we know that it is a complete projective resolution.

Theorem 2:

1. if n/ m then m- SG projective (R)∩ n- SG projective (R) = n – SG projective (R)
2. if n∤ m and m = np+k where p is a positive integer and 0< k < n then
   m- SG projective (R) ∩ n- SG projective (R)⊆ j-SG projective (R)

Proof: 1 it is trival since n/m

3. by above m- SG projective (R) ∩ n- SG projective (R)⊆m- SG projective (R) ∩ np- SG projective (R). M ∈m- SG projective (R) ∩ np - SG projective (R)

Then there exist an exact sequence

0→ $M \overset{f_m}{\to} P_{m-1} \overset{f_{m-1}}{\longrightarrow} \dots\dots . . P_1 \overset{f_1}{\to} P_0 \overset{f_0}{\to} M \to 0$.............................(1)

In Mod R with P_i projective for any 0≤ $i \le m-1$ put Li=Ker (Pi-1→ Pi-2) for any 2≤ $i \le m$ because M∈ pn- SG projective (R). It is easy to see that M and L_{kn} are projectivelyequivalent, that is there exist projective modules P and Q in Mod R, such that

M⊕P≅Q⊕L_{kn}

Proposition 2: For any n≥1 n-SG projective (R) is closed under direct sums.

Proof: let {Mj}j∈J be a family of n-SG projective modules in Mod R then for any j∈J there exist an exact sequence

0→ $\oplus_{j\in J} Mj \overset{f_n}{\to} \oplus_{j\in J} P_{n-1}^j \overset{f_{n-1}}{\longrightarrow} \dots\dots . . \oplus_{j\in J} P_1^j \overset{f_1}{\to} \oplus_{j\in J} P_0^j \overset{f_0}{\to} Mj \to 0$ in Mod R
because$\oplus_{j\in J} P_{n-1}^j$............$\oplus_{j\in J} P_0^j$ are projective and the obtained exact sequence is still exact after applying the functor HomR(-, P) when P∈Mod R is projective $\oplus_{j\in J}$Mj is n-SG projective.

Proposition 3: For any module M, the following are equivalent:

1. M is n-Strongly Gorenstien Projective
2. There exist a short sequence 0→ $M \xrightarrow{f_n} P_{m-1} \xrightarrow{f_{n-1}} \ldots\ldots..P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ where P is a projective module and
$Ext_1^n(M,Q)$ = 0 for any projective module Q
3. There exist a short exact sequence 0→ $M \xrightarrow{f_n} P_{m-1} \xrightarrow{f_{n-1}} \ldots\ldots..P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ where P is a projective module such that for any projective module Q the short sequence
0→ $Hom(M,Q) \xrightarrow{f_n} Hom(P_{n-1,}Q) \xrightarrow{f_{n-1}} \ldots\ldots..Hom(P_{1,}Q) \xrightarrow{f_1} Hom(P_0,Q) \xrightarrow{f_0} Hom(M,Q) \to 0$
is exact

Theorem 3: If a module is M is n-Strongly Gorenstein flat then it is a direct summand of a n-Strongly Gorenstein flat modules.

Proof: Similar to Theorem 1.

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